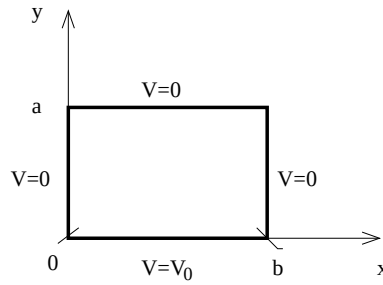


1. A two-dimensional electrostatic system is given in the following picture.



Solve the Laplace's equation and find the potential $V(x, y)$ for this system.

Solution. From the picture given, V depends on x and y only. Laplace's equation;

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Separation of variable; let

$$V(x, y) = X(x)Y(y),$$

then

$$\begin{aligned} X''Y + Y''X &= 0 \\ \frac{X''}{X} + \frac{Y''}{Y} &= 0. \end{aligned}$$

Necessarily $-\frac{X''}{X} = \frac{Y''}{Y} = \lambda$, where λ is the separation constant.

$$\begin{aligned} X'' + \lambda X &= 0 \\ Y'' - \lambda Y &= 0. \end{aligned}$$

For $0 < x < b$, $0 < y < a$, $X(x) \neq 0$ and $Y(y) \neq 0$;

$$\begin{aligned} V(0, y) &= X(0)Y(y) = 0 \Rightarrow X(0) = 0 \\ V(b, y) &= X(b)Y(y) = 0 \Rightarrow X(b) = 0 \\ V(x, 0) &= X(x)Y(0) = V_0 \Rightarrow \text{inseparable} \\ V(x, a) &= X(x)Y(a) = 0 \Rightarrow Y(a) = 0. \end{aligned}$$

$\lambda = 0$;

$$\begin{aligned} X'' &= 0 \\ X &= Ax + B \\ X(0) &= 0 \Rightarrow B = 0 \\ X(b) &= 0 \Rightarrow A = 0 \end{aligned}$$

Hence $X(x) = 0$, which is a trivial solution, therefore $\lambda \neq 0$. $\lambda < 0$; let $\lambda = -\alpha^2$. Then $X'' - \alpha^2 X = 0$.

$$\begin{aligned} (D^2 - \alpha^2)X &= 0 \\ (D - \alpha)(D + \alpha)X &= 0 \\ DX &= \pm \alpha X. \end{aligned}$$

$DX = \alpha X$;

$$\begin{aligned} \frac{dX}{X} &= \alpha dx \\ \int \frac{dX}{X} &= \int \alpha dx \\ \ln X &= \alpha x + \ln A_1 \\ X &= A_1 e^{\alpha x} \end{aligned}$$

$DX = -\alpha x$;

$$\begin{aligned} X &= A_2 e^{-\alpha x} \\ X(x) &= A_1 e^{\alpha x} + A_2 e^{-\alpha x} \end{aligned}$$

$$e^{\pm\alpha x} = \cosh \alpha x \pm \sinh \alpha x;$$

$$\begin{aligned} X &= A_1(\cosh \alpha x + \sinh \alpha x) + A_2(\cosh \alpha x - \sinh \alpha x) \\ &= (A_1 + A_2) \cosh \alpha x + (A_1 - A_2) \sinh \alpha x \\ &= B_1 \cosh \alpha x + B_2 \sinh \alpha x, \end{aligned}$$

where $B_1 = A_1 + A_2$ and $B_2 = A_1 - A_2$.

$$\begin{aligned} X(0) = 0 &\Rightarrow 0 = B_1 \\ X(b) = 0 &\Rightarrow 0 = B_2 \sinh \alpha b \end{aligned}$$

Since $\alpha \neq 0$ and $b \neq 0$, therefore $\sinh \alpha x \neq 0$ and consequently $B_2 = 0$. From this we have $X(x) = 0$, which is a trivial solution and therefore $\lambda \not\leq 0$.

$\lambda > 0$; let $\lambda = \beta^2$, then

$$\begin{aligned} X'' + \beta^2 X &= 0 \\ (D^2 + \beta^2)X &= 0 \\ DX &= \pm j\beta X \\ X &= C_0 e^{j\beta x} + C_1 e^{-j\beta x} \end{aligned}$$

$$e^{\pm j\beta x} = \cos \beta x \pm j \sin \beta x;$$

$$\begin{aligned} X &= C_0(\cos \beta x + j \sin \beta x) + C_1(\cos \beta x - j \sin \beta x) \\ &= (C_0 + C_1) \cos \beta x + j(C_0 - C_1) \sin \beta x \\ &= g_0 \cos \beta x + g_1 \sin \beta x \end{aligned}$$

where $g_0 = C_0 + C_1$ and $g_1 = j(C_0 - C_1)$.

$$\begin{aligned} X(0) = 0 &\Rightarrow 0 = g_0 \\ X(b) = 0 &\Rightarrow 0 = g_1 \sin \beta b \end{aligned}$$

if $g_1 \neq 0$;

$$\sin \beta b = 0 = \sin n\pi$$

Since $\beta \neq 0$ and $b \neq 0$, therefore $n \neq 0$. It follows that $\beta = \frac{n\pi}{b}$, where $n = 1, 2, 3, \dots$.

$$X_n = g_n \sin \frac{n\pi x}{b}$$

$$\lambda = \beta^2 = \frac{n^2 \pi^2}{b^2}$$

$$Y'' - \beta^2 Y = 0$$

$$Y = h_0 \cosh \beta y + h_1 \sinh \beta y$$

$$Y(a) = 0;$$

$$0 = h_0 \cosh \beta a + h_1 \sinh \beta a$$

$$h_1 = -h_0 \coth \beta a$$

$$Y = h_0 \cosh \beta y - h_0 \coth \beta a \sinh \beta y$$

$$Y_n = h_n \left(\cosh \frac{n\pi y}{b} - \coth \frac{n\pi a}{b} \sinh \frac{n\pi y}{b} \right)$$

$$V_n = g_n h_n \sin \frac{n\pi x}{b} \left(\cosh \frac{n\pi y}{b} - \coth \frac{n\pi a}{b} \sinh \frac{n\pi y}{b} \right)$$

$$V = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \left(\cosh \frac{n\pi y}{b} - \coth \frac{n\pi a}{b} \sinh \frac{n\pi y}{b} \right)$$

where $c_n = g_n h_n$.

$$V(x, 0) = V_0 = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \left(1 - \coth \frac{n\pi a}{b} (0) \right)$$

$$\int_0^b V_0 \sin \frac{n\pi x}{b} dx = \sum_{n=1}^{\infty} c_n \int_0^b \sin \frac{n\pi x}{b} \sin \frac{n\pi x}{b} dx$$

$$\int_0^\pi \sin mx \sin nx dx = \begin{cases} 0 & , m \neq n \\ \pi/2 & , m = n \end{cases}, \text{ therefore } m = n;$$

$$\begin{aligned} \int_0^b V_0 \sin dx &= c_n \int_0^b \sin^2 dx \\ -V_0 \frac{b}{n\pi} \left(\cos \frac{n\pi x}{b} \right) \Big|_0^b &= \frac{c_n}{2} \left(x - \left(\frac{b}{2n\pi} \right) \sin \frac{2n\pi x}{b} \right) \Big|_0^b \end{aligned}$$

$$\int \sin^2 ax dx = \frac{x}{2} - \frac{\sin 2ax}{4a} + C;$$

$$\begin{aligned} \frac{V_0 b}{n\pi} (1 - \cos n\pi) &= \frac{c_n}{2} \left(b - \frac{b}{2n\pi} \sin 2n\pi \right) \\ &= \frac{bc_n}{2} \left(1 - \frac{\sin 2n\pi}{2n\pi} \right) \\ &= \frac{bc_n}{2} \left(\frac{2n\pi - \sin 2n\pi}{2n\pi} \right) \end{aligned}$$

$$\begin{aligned} V_0 (1 - \cos n\pi) &= \frac{c_n}{4} (2n\pi - \sin 2n\pi) \\ c_n &= \frac{4V_0 (1 - \cos n\pi)}{(2n\pi - \sin 2n\pi)} \\ &= \begin{cases} \frac{8V_0}{2n\pi} & , n \text{ odd} \\ 0 & , n \text{ even} \end{cases} \\ &= \begin{cases} \frac{4V_0}{n\pi} & , n \text{ odd} \\ 0 & , n \text{ even} \end{cases} \end{aligned}$$

(n odd gives $\cos n\pi = -1$ and $\sin n\pi = 0$, while n even gives $\cos n\pi = 1$ and $\sin n\pi = 0$)

($\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$)

$$\begin{aligned} V &= \sum_{n=1,3,5}^{\infty} \left(\frac{4V_0}{n\pi} \right) \sin \frac{n\pi x}{b} \left(\cosh \frac{n\pi y}{b} - \coth \left(\frac{n\pi a}{b} \right) \sinh \frac{n\pi y}{b} \right) \\ &= \frac{4V_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{b} \left(\cosh \frac{n\pi y}{b} - \coth \left(\frac{n\pi a}{b} \right) \sinh \frac{n\pi y}{b} \right) \\ &= \frac{4V_0}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\sin \frac{n\pi x}{b} \sinh \frac{n\pi}{b} (a - y)}{\sinh \frac{n\pi a}{b}} \end{aligned}$$

#